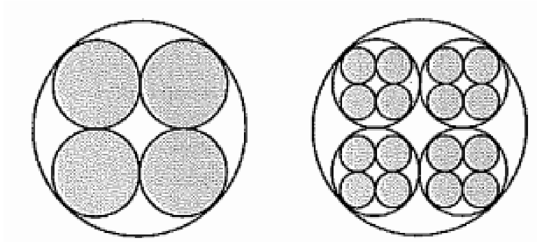
Problem of the week due Oct 26 in Prof. Kallenbach's mailbox

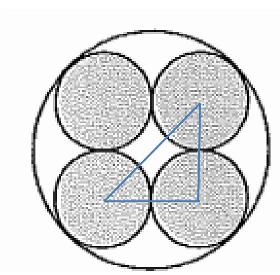


In Figure 1, 4 congruent circles are placed inside a larger circle so that each of the smaller ones is tangent to the large circle and to two other smaller ones. In Figure 2, each shaded circle in Figure 1 is replaced by a figure that is geometrically similar to Figure 1. Find the ratio of the shaded region of figure 2 to the shaded region of Figure 1. (Source – GRE practice exam)

A solution:

We'll examine only a single medium-sized circle from Figure 2, since the ratio of grey-to-total on one of those is the same as the one being asked for.

I will let r denote the radius of one of the shaded circles. Then the area of one shaded circle is πr^2, and the area of the entire shaded region is $4\pi r^2$. Since a single medium sized circle is an exact replica of a shaded circle from figure 1, all we need to do is figure out its area.



To do this, form an isosceles right triangle by joining the centers of two gray circles that are vertical from each other, then connecting the center of the lower one to the center of the one next to it, and then forming the hypotenuse which I call R. The length of each leg of the triangle is $2r$, and so the length of R is $R=\left(2\sqrt{2}\right)r$. No notice that the diameter of the outside circle is R + r + r, or $2\sqrt{2}r+2r=2r\left(\sqrt{2}+1\right)$.

Thus the radius of the outside circle is $r\left(\sqrt{2}+1\right)$, and its area is $\pi\left(r(\sqrt{2}+1)\right)^2=\pi r^2\left(\sqrt{2}+1\right)^2$. So finally the ratio asked for the original problem is

$$\frac{4\pi r^2}{\pi r^2\left(\sqrt{2}+1\right)^2}=\frac{4}{\left(\sqrt{2}+1\right)^2}$$