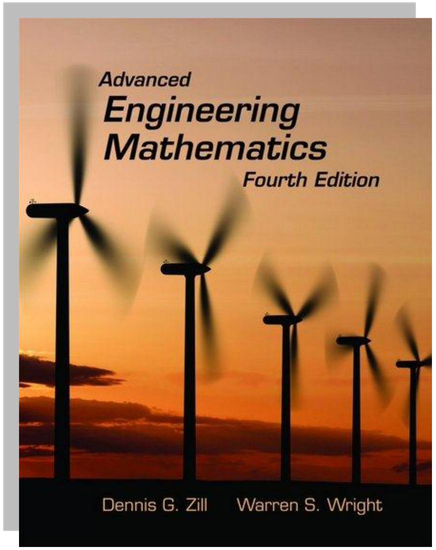
CHAPTER 5

Series Solutions of Linear Differential Equations



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Chapter Contents

5.1 Solutions about Ordinary Point

Review of Power Series

Recall from that a power series in $x - a$ has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

Such a series is said to be a **power series centered at *a*.**

❖ Convergence

$$\lim_{N\to\infty} S_N(x) = \lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n \quad \text{exists.}$$

❖ Interval of Convergence

The set of all real numbers for which the series converges.

❖ Radius of Convergence

If R is the radius of convergence, the power series converges for $|x - a| < R$ and
diverges for $|x - a| > R$.

❖ Absolute Convergence

Within its interval of convergence, a power series converges absolutely. That is, the following converges.

$$\sum_{n=0}^{\infty} |c_n (x-a)^n|$$

❖ Ratio Test

Suppose $c_n \neq 0$ for all n, and

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = |x-a|\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = L$$

If $L < 1$, this series converges absolutely, if $L > 1$, this series diverges, if $L = 1$, the test is inclusive.

❖ A Power Defines a Function

Suppose $y = \sum_{n=0}^{\infty} c_n x^n$

then

$$y' = \sum_{n=0}^{\infty} c_n n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} c_n n(n-1) x^{n-2} \tag{1}$$

❖ Identity Property

If all $c_n = 0$, then the series = 0.

❖ Analytic at a Point

A function *f* is analytic at a point *a*, if it can be represented by a power series in *x* – *a* with a positive radius of convergence. For example:

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots, \ \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad (2)$$

❖ Arithmetic of Power Series

Power series can be combined through the operations of addition, multiplication and division.

$$e^x \sin x$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{5040} - \cdots\right)$$

$$= (1)x + (1)x^2 + \left(-\frac{1}{6} + \frac{1}{2}\right)x^3 + \left(-\frac{1}{6} + \frac{1}{6}\right)x^4 + \left(\frac{1}{120} - \frac{1}{12} + \frac{1}{24}\right)x^5 + \cdots$$

$$= x + x^2 + \frac{x^3}{3} - \frac{x^5}{30} - \cdots$$

Example 1 Adding Two Power Series

Write $\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$ as one power series.

Solution:

Since

series starts with x for $n = 3$ ↓ series starts with x for $n = 0$ ↓

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} = 2\cdot 1c_2 x^0 + \sum_{n=3}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

we let $k = n - 2$ for the first series and $k = n + 1$ for the second series,

Example 1 (2)

then we can get the right-hand side as

same

$$2c_2 + \sum_{k=1}^{\infty}(k+2)(k+1)c_{k+2}x^k + \sum_{k=1}^{\infty}c_{k-1}x^k \qquad (3)$$

same

We now obtain

$$\sum_{n=2}^{\infty}n(n-1)c_n x^{n-2} + \sum_{n=0}^{\infty}c_n x^{n+1}$$

$$= 2c_2 + \sum_{k=1}^{\infty}[(k+2)(k+1)c_{k+2} + c_{k-1}]x^k \qquad (4)$$