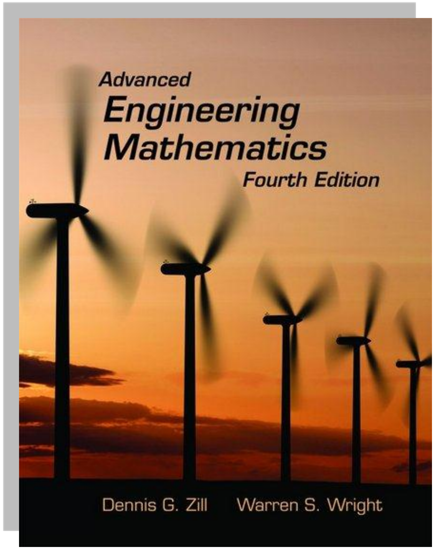
CHAPTER 5

Series Solutions of Linear Differential Equations



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Chapter Contents

A Solution

- Suppose the linear DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (5)$$

is put into

$$y'' + P(x)y' + Q(x)y = 0 \qquad (6)$$

Definition 5.1.1 Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of (5) if both $P(x)$ and $Q(x)$ in (6) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point**.

Polynomial Coefficients

- Since $P(x)$ and $Q(x)$ in (6) is a rational function,

$$P(x) = a_1(x)/a_2(x), \quad Q(x) = a_0(x)/a_2(x)$$

It follows that $x = x_0$ is an ordinary point of (5) if $a_2(x_0) \neq 0$.

Theorem 5.1.2 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of (5), we can always find two linearly independent solutions in the form of power series centered at x_0, that is,

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

A series solution converges at least of some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

Example 2 Power Series Solutions

Solve $y'' + xy = 0$

Solution:

We know there are no finite singular points.

Now, $y = \sum_{n=0}^{\infty} c_n x^n$ and $y'' = \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2}$

then the DE gives

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + x\sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} \qquad (7)$$

Example 2 (2)

From the result given in (4),

$$y'' + xy = 2c_2 + \sum_{k=1}^{\infty}[(k+1)(k+2)c_{k+2} + c_{k-1}]x^k = 0 \qquad (8)$$

Since (8) is identically zero, it is necessary all the coefficients are zero, $2c_2 = 0$, and

$$(k+1)(k+2)c_{k+2} + c_{k-1} = 0\,, \quad k = 1, 2, 3, \cdots \qquad (9)$$

Now (9) is a **recurrence relation**, since $(k + 1)(k + 2) \neq 0$, then from (9)

$$c_{k+2} = -\frac{c_{k-1}}{(k+1)(k+2)}\,, \quad k = 1, 2, 3, \cdots \qquad (10)$$

Example 2 (3)

Thus we obtain

$$k = 1, \quad c_3 = -\frac{c_0}{2 \cdot 3}$$

$$k = 2, \quad c_4 = -\frac{c_1}{3 \cdot 4}$$

$$k = 3, \quad c_5 = -\frac{c_2}{4 \cdot 5} = 0 \qquad \leftarrow c_2 \text{ is zero}$$

$$k = 4, \quad c_6 = -\frac{c_3}{5 \cdot 6} = \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} c_0$$

$$k = 5, \quad c_7 = -\frac{c_4}{6 \cdot 7} = \frac{1}{3 \cdot 4 \cdot 6 \cdot 7} c_1$$

Example 2 (4)

$$k = 6, \qquad c_8 = -\frac{c_5}{7 \cdot 8} = 0 \quad \leftarrow c_5 \text{ is zero}$$

$$k = 7, \qquad c_9 = -\frac{c_6}{8 \cdot 9} = -\frac{1}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9} c_0$$

$$k = 8, \qquad c_{10} = -\frac{c_7}{9 \cdot 10} = -\frac{1}{3 \cdot 4 \cdot 6 \cdot 7 \cdot 9 \cdot 10} c_1$$

$$k = 9, \qquad c_{11} = -\frac{c_8}{10 \cdot 11} = 0 \quad \leftarrow c_8 \text{ is zero}$$

and so on.

Example 2 (5)

Then the power series solutions are

$$y = c_0 y_1 + c_1 y_2$$

$$y = c_0 + c_1 x + 0 - \frac{c_0}{2 \cdot 3} x^3 - \frac{c_1}{3 \cdot 4} x^4 + 0 + \frac{c_0}{2 \cdot 3 \cdot 5 \cdot 6} x^6$$

$$+ \frac{c_1}{3 \cdot 4 \cdot 6 \cdot 7} x^7 + 0 + \ldots.$$

Example 2 (6)

$$y_1(x) = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \cdots$$

$$= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{2\cdot 3\cdots(3n-1)(3n)}x^{3k}$$

$$y_2(x) = 1 - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \cdots$$

$$= x + \sum_{k=1}^{\infty} \frac{(-1)^k}{3\cdot 4\cdots(3n)(3n+1)}x^{3k+1}$$

Example 3 Power Series Solution

Solve $(x^2+1)y''+xy'-y=0$

Solution:

Since $x^2 + 1 = 0$, then $x = i, -i$ are singular points. A power series solution centered at 0 will converge at least for $|x| < 1$. Using the power series form of y, y' and y'', then

$$(x^2+1)\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + x\sum_{n=1}^{\infty} nc_n x^{n-1} - \sum_{n=0}^{\infty} c_n x^n$$

$$= \sum_{n=2}^{\infty} n(n-1)c_n x^n + \sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^n$$

Example 3 (2)

$$= 2c_2x^0 - c_0x^0 + 6c_3x + c_1x - c_1x + \underbrace{\sum_{n=2}^{\infty} n(n-1)c_nx^n}_{k=n}$$

$$+ \underbrace{\sum_{n=4}^{\infty} n(n-1)c_nx^{n-2}}_{k=n-2} + \underbrace{\sum_{n=2}^{\infty} nc_nx^n}_{k=n} - \underbrace{\sum_{n=2}^{\infty} c_nx^n}_{k=n}$$

$$= 2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty}[k(k-1)c_k + (k+2)(k+1)c_{k+2} + kc_k - c_k]x^k$$

$$= -2c_2 - c_0 + 6c_3x + \sum_{k=2}^{\infty}[(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2}]x^k = 0$$

Example 3 (3)

From the above, we get $2c_2 - c_0 = 0$, $6c_3 = 0$, and

$$(k+1)(k-1)c_k + (k+2)(k+1)c_{k+2} = 0$$

Thus $c_2 = c_0/2$, $c_3 = 0$, $c_{k+2} = (1 - k)c_k/(k + 2)$

Then

$$c_4 = -\frac{1}{4}c_2 = -\frac{1}{2 \cdot 4}c_0 = -\frac{1}{2^2 2!}c_0$$

$$c_5 = -\frac{2}{5}c_3 = 0$$ ← c_3 is zero

$$c_6 = -\frac{3}{6}c_4 = -\frac{3}{2 \cdot 4 \cdot 6}c_0 = \frac{1 \cdot 3}{2^3 3!}c_0$$

$$c_7 = -\frac{4}{7}c_5 = 0$$ ← c_5 is zero

Example 3 (4)

$$c_8 = -\frac{5}{8}c_6 = -\frac{3\cdot 5}{2\cdot 4\cdot 6\cdot 8}c_0 = -\frac{1\cdot 3\cdot 5}{2^4 4!}c_0$$

$$c_9 = -\frac{6}{9}c_7 = 0$$ ← c_7 is zero

$$c_{10} = -\frac{7}{10}c_8 = \frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 10}c_0 = \frac{1\cdot 3\cdot 5\cdot 7}{2^5\cdot 5!}c_0$$

and so on.

Example 3 (5)

Therefore,

$$
\begin{aligned}
y &= c_0 + c_1 x + c_2 x^2 + c_3 x^3 + c_4 x^4 + c_5 x^5 \\
&\quad + c_6 x^6 + c_7 x^7 + c_8 x^8 + c_9 x^9 + c_{10} x^{10} + \cdots \\
&= c_0 \left[1 + \frac{1}{2} x^2 - \frac{1}{2^2 2!} x^4 + \frac{1 \cdot 3}{2^3 3!} x^6 - \frac{1 \cdot 3 \cdot 5}{2^4 4!} x^8 + \frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 5!} x^{10} - \cdots \right] + c_1 x \\
&= c_0 y_1(x) + c_1 y_2(x)
\end{aligned}
$$

$$
y_1(x) = 1 + \frac{1}{2} x^2 + \sum_{n=2}^{\infty} (-1)^{n-1} \frac{1 \cdot 3 \cdot 5 \cdots (2n-3)}{2^n n!} x^{2n}, \quad |x| < 1
$$

$$
y_2(x) = x
$$

Example 4 Three-Term Recurrence Relation

If we seek a power series solution $y(x)$ for

$$y'' - (1 + x)y = 0$$

we obtain $c_2 = c_0/2$ and the recurrence relation is

$$c_{k+2} = \frac{c_k + c_{k-1}}{(k+1)(k+2)}, \quad k = 1, 2, 3, \cdots$$

Examination of the formula shows c_3, c_4, c_5, ... are expresses in terms of both c_1 and c_2. However it is more complicated. To simplify it, we can first choose $c_0 \neq 0$, $c_1 = 0$.

Example 4 (2)

Then we have

$$c_2 = \frac{1}{2}c_0$$

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_0}{2 \cdot 3} = \frac{1}{6}c_0$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_0}{2 \cdot 3 \cdot 4} = \frac{1}{24}c_0$$

$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_0}{4 \cdot 5}\left[\frac{1}{6} + \frac{1}{2}\right] = \frac{1}{30}c_0$$

and so on. Next, we choose $c_0 = 0$, $c_1 \neq 0$, then

$$c_2 = \frac{1}{2}c_0 = 0$$

Example 4 (3)

$$c_3 = \frac{c_1 + c_0}{2 \cdot 3} = \frac{c_1}{2 \cdot 3} = \frac{1}{6} c_1$$

$$c_4 = \frac{c_2 + c_1}{3 \cdot 4} = \frac{c_1}{3 \cdot 4} = \frac{1}{12} c_1$$

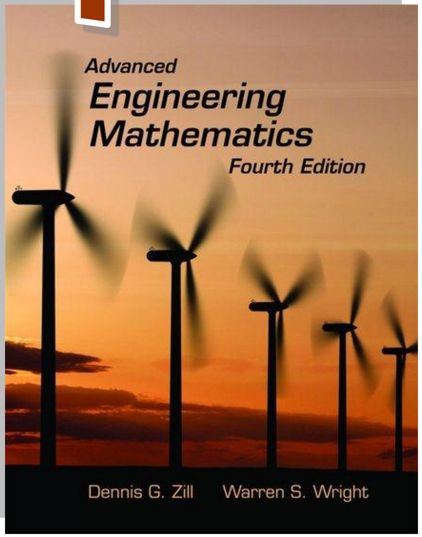
$$c_5 = \frac{c_3 + c_2}{4 \cdot 5} = \frac{c_1}{4 \cdot 5 \cdot 6} = \frac{1}{120} c_1$$

and so on. Thus we have $y = c_0 y_1 + c_1 y_2$, where

$$y_1(x) = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \frac{1}{30}x^5 + \cdots$$

$$y_2(x) = x + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{1}{120}x^5 + \cdots$$

Thank You！



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