Math 282 Winter 2016 Exam 2 80 pts Feb 22 Name________________________

1. Consider the 1st-order IVP $y' = \frac{1}{3} \cdot y \cdot (5-y); y(0) = 1$.

a. With step size h = 1.0, use Euler's method (**formula**) to find y_1, the Euler approximation at $x_1 = 1.0$.

$$y_1 = y_0 + \Delta x \cdot y_0 \cdot (5 - y_0) = 1.0 + \frac{1}{3}\left(1.0 \cdot 1.0 \cdot (5 - 1.0)\right) = 2.3333$$

b. Now use your TI Euler_Spreadsheet file to find the rest of the table values. Place your approximate y's in column 3. Use 3 decimals places or more.

I	x_i	y_i	$y_a(x_i)$	Error
0	0.00	1.000	1.0	
1	1.0	2.3333	2.8482	
2	2.0		4.3756	
3	3.0		4.8688	
4	4.0		4.9747	
5	5.0		4.9952	

c. Find the analytic solution to the IVP by treating it as a logistic equation. You may use formula on the last page, or solve using the separable method.

According to the formula, the logistic equation is defined by

$\frac{dQ}{dt} = k \cdot Q \cdot (M - Q)$. So in our case, $k = \frac{1}{3}, M = 5.0, Q_0 = 1.0$

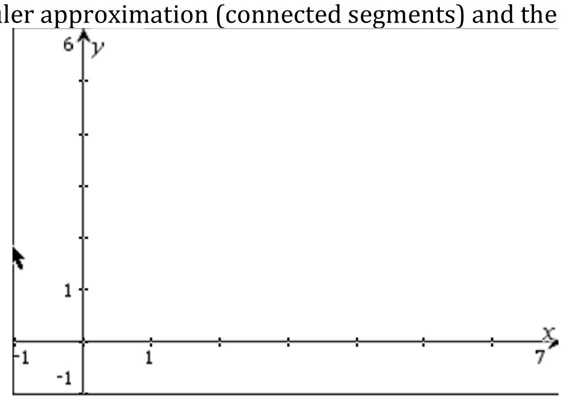
The solution according to the formula is given by $Q(t) = \frac{MQ_0}{Q_0 + (M - Q_0)e^{-Mkt}}$, or

$$\frac{5 \cdot 1.0}{1.0 + (5.0 - 1.0)e^{-5 \cdot \frac{1}{3} \cdot t}} = \frac{5}{1 + 4e^{-\frac{5}{3}t}}$$

d. Fill in column 4 of the table by plugging the x_i's into your answer from part b

e. Fill in column 5 of the table by computing the absolute value of the difference between the actual and the estimate

f. On these axes, roughly sketch the Euler approximation (connected segments) and the graph of the analytic solution.



2. **Do any 4 of these 5 problems.** If you try all 5 let me know which one not to grade.

a. A specimen of charcoal found at Stonehenge contains 63% as much Carbon-14 as a present-day sample of the same mass. If carbon-14 has a half-life of 5700 years, how old is the specimen?

The solution follows the exponential decay model of $Q'(t) = kQ(t); Q(t) = Q_0 e^{kt}$. Remember that $k = \frac{\ln 2}{\tau} = \frac{\ln 2}{5700} \approx .000122$. So in our problem, $Q(t) = Q_0 e^{-0.000122t}$

We need to find t such that $.63Q_0 = Q_0 e^{-.000122t}$, which means $t = \frac{\ln(.63)}{-0.000122} \approx 3787.18$ years

b. A roast, initially at $50^o F$ is placed in a $375^0 F$ oven at 5:00 PM. After 75 minutes, its internal temperature is $125^o F$. At what time will the roast be $150^o F$ (medium rare)?

Newton's Law Of Cooling says that $\frac{dT}{dt} = k(T - T_m)$, with solution $T(t) = T_m + (T_0 - T_m)e^{kt}$. In our case $T_0 = 50, T_m = 375, T(75) = 125$

So $T(t) = 375 + (50 - 375)e^{kt} = 375 - 325e^{kt}$

T(75) = 125, so $125 = 375 - 325e^{75k} \Rightarrow -250 = -325e^{75k} \Rightarrow k = \frac{1}{75}\ln\left(\frac{10}{13}\right) \approx -.0035$

$T(t) = 375 - 325e^{-.0035t}$. We set this equal to 150 and solve for t to get

$t = \frac{\ln\left(\frac{225}{325}\right)}{-.0035} \approx 105$ minutes

c. An object weighing 192 lbs. is dropped from a helicopter. The wind resistance during the fall is proportional to the speed at which the object is falling, with constant of proportionality k = 4.0 lb/ft². Find the terminal velocity.

d. A specimen that begins with 20 bacteria exhibits **logistic** growth in a medium with maximum population of M = 150, and growth constant k = .0004.

i. Find a formula for the number of bacteria after t days.

We can use the formula for the quantity, replacing Q0 with 20, M with 150, and k with .0004. This gives us $Q(t) = \frac{20 \cdot 150}{20 + (50 - 20)e^{-.0004 \cdot 50 \cdot t}} = \frac{3000}{20 + 30e^{-.02t}}$

ii. After how many days will the population reach 90% of its capacity?

90% of 150 is 135, so we solve $135 = \frac{3000}{20 + 30e^{-.02t}}$ for t to get t = 130.134 days

e. Suppose that a beginning population of $Q_0 = 900$ fish in a pond is attacked by a disease. The fish cease to reproduce (no more are born), and the death rate is proportional to $\frac{1}{\sqrt{Q}}$. After 6 weeks the original 900 fish were down to 441. After how many weeks did the fish population die off completely?

Formulas

Newton's Law Of Cooling: $\frac{dT}{dt} = k(T - T_m)\,; T = T_m + (T_0 - T_m)e^{kt}$

Logistic Growth Model: $\frac{dQ}{dt} = k \cdot Q \cdot (M - Q); Q(t) = \frac{MQ_0}{Q_0 + (M - Q_0)e^{-Mkt}}$

Exponential Unbounded Growth/Decay: $Q'(t) = kQ(t); Q(t) = Q_0 e^{kt}$

Motion: $\begin{cases} F = ma; \\ mv' = -mg \pm kv \Rightarrow v(t) = \left(v_0 + \frac{mg}{k}\right)e^{-kt} - \frac{mg}{k}. \\ mv' = -mg \pm kv^2 \end{cases}$

Gravity only: $h(t) = -\frac{g}{2}t^2 + v_0 t + h_0$

Euler's Method: $y_{n+1} = y_n + f(x_n, y_n) \cdot \Delta x$