Differential Equations

Chapter 08:
Series Solutions of Second Order Linear Equations

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8.1 Review of Power Series

DEFINITION 8.1.1 - A **power series** is an infinite series of the form

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots. \qquad (1)$$

The constants $a_0, a_1, a_2, \ldots$ are called the **coefficients** of the series, the constant x_0 is called the **center** of the series, and x is a variable. Setting $x_0 = 0$ in Eq. (1) gives us a **power series centered at *x*0 = 0**:

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots.$$

Convergence Concepts

DEFINITION 8.1.2 A power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is said to **converge** at a point *x* if the sequence of **partial sums**

$$S_m(x) = \sum_{n=0}^{m} a_n(x - x_0)^n$$
$$= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots + a_m(x - x_0)^m$$

converges as $m \to \infty$. The sum of the series at the point *x* is defined to be the limit of the sequence of partial sums, and we write

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = \lim_{m \to \infty} S_m(x).$$

If the limit of the sequence of partial sums does not exist, then the series is said to **diverge** at *x*.

Example

Use Definition 8.1.2 to show that the infinite **geometric series**

$$a + ax + ax^2 + \cdots = \sum_{n=0}^{\infty} ax^n$$

converges to $a/(1-x)$ if $|x| < 1$ and diverges if $|x| \geq 1$ provided that $a \neq 0$.

DEFINITION 8.1.3

The series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$ is said to **converge absolutely** at a point x if the series

$$\sum_{n=0}^{\infty} |a_n(x-x_0)^n| = \sum_{n=0}^{\infty} |a_n||x-x_0|^n \qquad (2)$$

converges.

Theorems

- THEOREM 8.1.4 (**absolute convergence implies convergence)**

If the power series $\sum_{n=0}^{\infty} |a_n(x-x_0)^n|$ converges at x, then so does the series $\sum_{n=0}^{\infty} a_n(x-x_0)^n$.

- THEOREM 8.1.5 (**The Ratio Test)**

The Ratio Test. If $a_n \neq 0$, and if, for a fixed value of x,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x-x_0)^{n+1}}{a_n(x-x_0)^n}\right| = |x-x_0| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-x_0|L,$$

then the power series converges absolutely at that value of x if $|x-x_0|L < 1$ and diverges if $|x-x_0|L > 1$. If $|x-x_0|L = 1$, the test is inconclusive.

Radius of Convergence.

If $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is a power series, then either

1. The series converges absolutely for all x, or

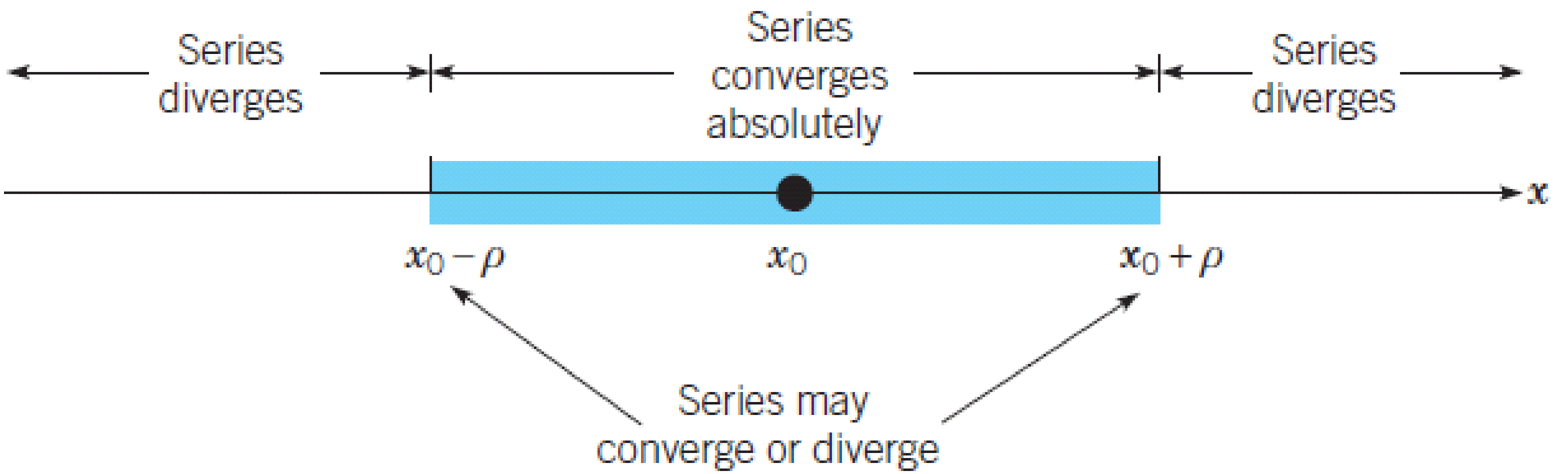
2. The series converges only for $x = x_0$, or

3. There exists a number $\rho > 0$ such that $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ converges absolutely for $| x - x_0 | < \rho$ and diverges for $| x - x_0 | > \rho$.

The number ρ in Case 3 is called the **radius of convergence** and the interval $| x - x_0 | < \rho$ is called the **interval of convergence**. This case is illustrated by the shaded region in Figure 8.1.1. We write $\rho = \infty$ in Case 1 and $\rho = 0$ in Case 2. Using these conventions, we can state that *each power series has a radius of convergence* ρ, *where* $0 \leq \rho \leq \infty$. If $0 < \rho < \infty$, *the series may either converge or diverge when* $|x - x_0| = \rho$.

FIGURE 8.1.1

- The interval of convergence of a power series if $0 < \rho < \infty$.



Example

- **Question**

Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(x+1)^n}{n2^n}.$$

- **Answer**
- The given power series converges for $-3 \leq x < 1$ and diverges otherwise. It converges absolutely for $-3 < x < 1$ and has a radius of convergence 2.

Algebraic Operations on Power Series

Addition and subtraction. The series can be added or subtracted termwise, and

$$f(x) \pm g(x) = \sum_{n=0}^{\infty} (a_n \pm b_n)(x - x_0)^n = a_0 \pm b_0 + (a_1 \pm b_1)(x - x_0) + \cdots.$$

The resulting series converges at least for $|x - x_0| < \rho$.

Multiplication. The series can be formally multiplied, and

$$f(x)g(x) = \left[\sum_{n=0}^{\infty} a_n (x - x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n (x - x_0)^n\right] = \sum_{n=0}^{\infty} c_n (x - x_0)^n,$$

where $c_n = a_0 b_n + a_1 b_{n-1} + \cdots + a_n b_0$. The resulting series converges at least for $|x - x_0| < \rho$.

The next theorem is a tool that is often used to help find the coefficients of a power series.

THEOREM 8.1.6

The Identity Principle. If $\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{n=0}^{\infty} b_n(x-x_0)^n$ for each x in some open interval with center x_0, then $a_n = b_n$ for $n = 0, 1, 2, 3, \ldots$. In particular, if $\sum_{n=0}^{\infty} a_n(x-x_0)^n = 0$ for each such x, then $a_0 = a_1 = \cdots = a_n = \cdots = 0$.

Division. If $g(x_0) \neq 0$, the series can be formally divided, and

$$\frac{f(x)}{g(x)} = \sum_{n=0}^{\infty} d_n(x-x_0)^n.$$

In most cases the coefficients d_n can be most easily obtained by equating coefficients in the equivalent relation

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \left[\sum_{n=0}^{\infty} d_n(x-x_0)^n\right]\left[\sum_{n=0}^{\infty} b_n(x-x_0)^n\right]$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} d_k b_{n-k}\right)(x-x_0)^n.$$

Using Theorem 8.1.6, we then set

$$a_n = \sum_{k=0}^{n} d_k b_{n-k} \quad \text{for each } n = 0, 1, 2, \ldots.$$

This gives

$$a_0 = d_0 b_0, \qquad a_1 = d_0 b_1 + d_1 b_0, \qquad a_2 = d_0 b_2 + d_1 b_1 + d_2 b_0, \qquad \ldots,$$

which can be solved sequentially for $d_0, d_1, d_2, \ldots$. In the case of division the radius of convergence of the resulting power series may be less than ρ. For example, $\rho = \infty$ for $f(x) = 1$ and $g(x) = 1 - x$, but $\rho = 1$ for $f(x)/g(x) = 1/(1-x) = \sum_{n=0}^{\infty} x^n$.

Taylor Series and Analytic Functions

- **DEFINITION 8.1.7** Let *f* be a function with derivatives of all orders throughout the interval $|x - x_0| < \rho$, where $\rho > 0$. Then the **Taylor series of *f* at** x_0 is the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0) + \cdots.$$

- **DEFINITION 8.1.8** A function *f* that has a power series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

with a radius of convergence $\rho > 0$ is said to be **analytic** at *x*0.

Shift of Index of Summation

- Shifting the index of summation in a power series is analogous to changing the variable of integration in an integral. This operation, in conjunction with Theorem 8.1.6, is a useful tool for computing power series solutions of differential equations.
- **EXAMPLE**

Write $\sum_{n=2}^{\infty} a_n x^n$ as a series whose first term corresponds to $n = 0$ rather than $n = 2$. Let $m = n - 2$; then $n = m + 2$, and $n = 2$ corresponds to $m = 0$. Hence

$$\sum_{n=2}^{\infty} a_n x^n = \sum_{m=0}^{\infty} a_{m+2} x^{m+2}.$$